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Median graphs and hypercubes, some new characterizations

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Abstract

A projection (antiprojection respectively) of a vertex x of a graph G over a subset S of vertices is a vertex of S at a minimal (maximal respectively) distance from x. Which graphs are such that there is uniqueness of the antiprojection or uniqueness of the projection of a vertex over intervals or convex sets? We study these four properties and obtain new characterizations of hypercubes and median graphs. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs in this paper are assumed to be simple, finite and connected. We denote by $d(u,v)$ the length of a shortest (u,v)*-path* for any vertices u,v of a graph G. The level decomposition of G from a vertex u is the partition of the vertices of G in $N_0,N_1,\dots,N_k$ such that $N_i=\{x/d(u,x)=i\}$. In such a decomposition edges joint vertices in consecutive levels or in the same level. For any two vertices u and v of G the interval between u and v is the set: $I(u,v)=\{w\in V(G)/w$ *lies on a shortest* (u,v)*-path*$\}$. A set of vertices C is said to be convex if for any two elements u and v of C the interval $I(u,v)$ is contained in C. The projection (antiprojection respectively) of a vertex x over a subset S of vertices is the set $P(x,S)$ ($\mathrm{AP}(x,S)$ respectively) of vertices of S at minimal (maximal respectively) distance from x.

In the hypercube intervals are the only convex sets and induce hypercubes of lower dimensions. For a given property of the hypercube an interesting problem is to study graphs with the same property. For example in this graph the projection or antiprojection of any vertex over a subhypercube is a unique vertex. Thus we will study graphs satisfying one of the following properties:

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(P1) for all u, v and w: $|\mathrm{AP}(w, I(u, v))| = 1$;
(P2) for all convex C and all vertex w, we have: $|\mathrm{AP}(w, C)| = 1$;
(P3) for all u, v and w: $|P(w, I(u, v))| = 1$;
(P4) for all convex C and all vertex w. We have: $|P(w, C)| = 1$.

Proposition 1. *Let G satisfying any of* (P1), (P2), (P3) *or* (P4) *then G is bipartite.*

Proof. Assume G is not bipartite and consider an odd cycle $u_1, u_2 \dots u_{p+1} u_{p+2} \dots u_{2p+1} u_1$ of minimal length $2p + 1$. The set $\{u_{p+1}, u_{p+2}\}$ is convex and is also the interval $I(u_{p+1}, u_{P+2})$. Furthermore, $d(u_1, u_{p+1}) = d(u_1, u_{p+2}) = p$ in contradiction with any of the four properties. □

Proposition 2. *If G satisfies* (P1) *or* (P3) *then G is without* $K_{2,3}$ *as an induced subgraph.*

Proof. Assume (P1) and let x, y, z be three common neighbours of u and v. The graph G is bipartite and vertices in $I(u, v)$ are u, v and the common neighbours of u and v, thus are at maximal distance 2 of x. But $d(x, y) = d(x, z) = 2$ and y, z are both in $\mathrm{AP}(x, I(u, v))$ in contradiction with (P1).

Assume now (P3) and let x, y, z be three common neighbours of u and v. The graph G is bipartite thus $\{y, u, v, z\} \subset I(y, z)$ and $x \notin I(y, z)$. But $d(x, u) = d(x, v) = 1$ and u, v are both in $P(x, I(y, z))$ contradicting (P3). □

2. Properties (P1), (P2) and (0,2)-graphs

The notion of $(0, 2)$-graph has been introduced by Mulder [6]. A connected graph is a $(0, 2)$-graph if any two distinct vertices have exactly two common neighbours or none at all.

Proposition 3. *If G satisfies* (P1) *then G is a* $(0, 2)$*-graph.*

Proof. Assume w is a common neighbour of u and v. Then u and v have an other common neighbour t else $I(u, v) = \{u, v, w\}$ and $\mathrm{AP}(w, I(u, v)) = \{u, v\}$. By Proposition 2, t is unique. □

Proposition 4. *If G satisfies* (P1) *and* $N_0, N_1, \dots, N_k$ *is a level decomposition from a vertex x of G then every* 4*-cycle intersects exactly three levels.*

Proof. G is bipartite thus every 4-cycle intersects two or three levels. Assume that there exists a 4-cycle $uvwtu$ interesting only the two levels N_i and N_{i+1} with u in N_i then again because G is bipartite we have v, t in N_{i+1} and w in N_i. But $I(u, w) = \{u, v, t, w\}$ and thus $\mathrm{AP}(x, I(u, w)) = \{v, t\}$ contradicting (P1). □

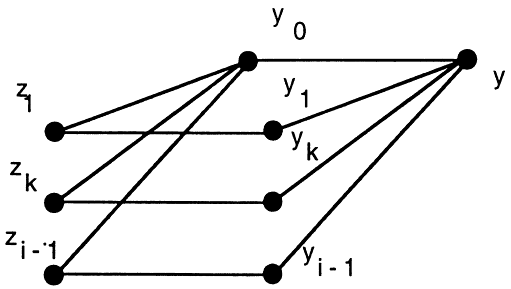

Fig. 1. Median graphs, (0,2)-graphs and hypercubes.

Hypercubes are characterized among $(0,2)$-graphs by the following theorem [4,6]:

Theorem 5. *Let G be a* $(0,2)$*-graph. Then G is regular. Let d be the degree, then* $|V(G)|\leqslant 2^d$ *with equality if and only if G is the hypercube* Q_d.

From this characterization of hypercubes we can deduce the corollary:

Lemma 6. *Let G be a* $(0,2)$*-graphs with a level decomposition such that every* 4*-cycle intersects exactly* 3 *levels then G is a hypercube.*

Proof. Notice first that there are no edges between vertices in the same level and thus the graph is bipartite. Let $N_0, N_1, \ldots, N_k$ be the level decomposition from the vertex x and let d be the degree of G. By induction on i we will prove that $|N_i| = \binom{d}{i}$ and that every vertex y of N_i has exactly i neighbours in N_{i-1}. This is clear for $i=0$ and 1, assume the property for $j<i$. Let y be in N_i and y_0 a neighbour of y in N_{i-1} (see Fig. 1). By induction hypothesis y_0 has exactly $i-1$ neighbours in N_{i-2} say $z_1, z_2, \ldots, z_{i-1}$. For $k=1$ to $i-1$ the 2-path y, y_0, z_k is closed by a vertex y_k, neighbour of y in N_{i-1}. The y_k are all distinct and it is clear that y has no other neighbours in N_{i-1} than $y_0, y_1, \ldots, y_{i-1}$ because a 4-cycle meets three levels therefore a path y_0, y, t with t in N_{i-1} is closed by a vertex in N_{i-2}, thus a z_k for some k. There are exactly $i|N_i|$ edges between N_i and N_{i-1}; but on the other hand this number is also $|N_{i-1}|(d-i+1)$ thus $|N_i| = \binom{d}{i}$ and $|V(G)| = 2^d$. □

Therefore form Lemma 6, and Propositions 3 and 4 we obtain:

Theorem 7. *G satisfies* (P1) *if and only if G is a hypercube*

A similar result holds for (P2):

Theorem 8. *Let G be a graph without* $K_{2,3}$ *then G satisfies* (P2) *if and only if G is a hypercube.*

Proof. Assume that G is without $K_{2,3}$ and satisfies (P2) and let w be a common neighbour of u and v. Then u and v have an other common neighbour t else $\{u,v,w\}$ is a convex and $\mathrm{AP}(w,\{u,v,w\})=\{u,v\}$. $K_{2,3}$ are forbidden thus t is unique and G is a $(0,2)$-graph.

Assume that $uvwtu$ is a 4-cycle intersecting only two levels N_i and N_{i+1} in a level decomposition from x with u in N_i. Then v,t is in N_{i+1} and w in N_i. $I(u,w)=\{u,v,w,t\}$ is a convex and $\mathrm{AP}(x,\{u,v,w,t\}) = \{t,v\}$. A contradiction. Then we can use our Lemma 6. □

Theorem 5 shows that hypercubes are $(0,2)$-graphs of maximal order. Using a computer one of the authors produced a table of the 33 (0,2)-graphs of order less than 32 [5]. Many interesting open problems are linked to this class of graphs [2].

3. Properties (P3), (P4) and median graphs

For every vertices u,v and w let $I(u,v,w)$ be the set $I(u,v)\cap I(v,w)\cap I(w,u)$. A median graph is a graph such that for all u,v,w we have $|I(u,v,w)|=1$. Several characterizations of median graphs have appeared. The following one is due to Mulder [7]:

Theorem 9. *Let G be a connected triangle free graph. If $|I(u,v,w)|=1$ for any three vertices u,v,w such that $d(u,v)=2$ then G is a median graph.*

Berrachedi [1] proved the following results:

Proposition 10. *For every graph G and any vertices u,v,w of G we have $I(u,v,w)\subset (P(w,I(u,v))$. Furthermore, if $I(u,v,w)\neq\emptyset$ then $I(u,v,w)=P(w,I(u,v))$.*

Proposition 11. *If G satisfies* (P3) *then $I(u,v,w)\neq\emptyset$ for any three vertices u,v,w such that* $\mathrm{d}(u,v)=2$.

Taking advantage of these, together with Theorem 9 we get the characterization below.

Theorem 12. *G satisfies* (P3) *if and only if G is a median graph.*

A similar result holds for (P4):

Theorem 13. *G satisfies* (P4) *and is without $K_{2,3}$ if and only if G is a median graph.*

Proof. Notice first that median graphs are bipartite and without $K_{2,3}$ (by a direct proof or using Theorem 12 and Propositions 1 and 2). Assume that G is a median graph and does not satisfy (P4); then there exist a convex C, u,v in C and w such that u,v are in $P(w,C)$. But $I(u,v)\subseteq C$ and thus $\{u,v\}\subseteq P(w,I(u,v))$ contradicting Theorem 12.

Conversely, consider a nonmedian graph G satisfying (P4) and without $K_{2,3}$. Then there exist u,v,w such that $d(u,v)=2$ and $|I(u,v,w)|\neq 1$.

If $|I(u,v,w)|\geqslant 2$ then $I(u,v,w)=P(w,I(u,v))$ (Proposition 10). But G is without $K_{2,3}$, thus $I(u,v)$ is convex contradicting (P4).

If $|I(u,v,w)|=0$ let w' be the unique projection of w on the convex $I(u,v)$. Then w' must be equal to u or v else w' would be in $I(u,v,w)$. Assume $w'=u$. Then from $I(u,v,w)=\emptyset$ we have $u\notin I(v,w)$ thus $d(w,v)<d(w,u)+2$. But from $\{u\}=P(w,I(u,v))$ we deduce $d(w,u)\leqslant d(w,v)$, thus $d(w,v)=d(w,u)+1$ and G contains an odd cycle, a contradiction with (P4). □

Ivan Havel [3] exhibited examples of graphs satisfying (P2) or (P4) but with $K_{2,3}$ as an induced subgraph. Consider the graph obtained from $K_{n,n}$ $(n>4)$ by deletion of a perfect matching. The only convex sets are a vertex, two adjacent vertices or all the vertices. This graph satisfies (P2) and (P4), but is not median and thus we cannot suppress the condition without $K_{2,3}$ in Theorems 8 and 13.

References

[1] A. Berrachedi, A new characterization of median graphs, Discrete Math. 128 (1994) 385–387.
[2] A. Berrachedi, M. Mollard, On two problems about (0,2)-graphs and interval-regular graphs, Ars Combin. 49 (1998) 303–309.
[3] I. Havel, Personal communication.
[4] J.M. Laborde, S.P. Rao Hebbare, Another characterization of hypercubes, Discrete Math. 39 (1982) 161–166.
[5] M. Mollard, Table of (0,2)-graphs of order less than 32, RR imag no. 939, Grenoble, June 1994.
[6] H.M. Mulder, $(0,\lambda)$-graphs and n-cubes, Discrete Math. 28 (1979) 179–188.
[7] H.M. Mulder, n-Cubes and median graphs, J. Graph Theory 4 (1980) 107–110.